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BARBARA A. VAN AMEROM

FOCUSING ON INFORMAL STRATEGIES WHEN LINKING ARITHMETIC TO EARLY ALGEBRA

ABSTRACT. In early algebra students often struggle with equation solving. Modeled on Streefland's studies of students' own productions a prototype pre-algebra learning strand was designed which takes students' informal (arithmetical) strategies as a starting point for solving equations. In order to make available the skills and tools needed for manipulating equations, the students are stimulated and guided to develop suitable algebraic language, notations and reasoning. One of the results of the study is that reasoning and symbolizing appear to develop as independent capabilities. For instance, students in grades 6 and 7 can solve equations at both a formal and an informal level, but formal symbolizing has been found to be a major obstacle.

KEY WORDS: arithmetic, early algebra, equations, history of mathematics, reasoning, symbolizing

INTRODUCTION

Many secondary school students experience great difficulty in learning how to solve equations. Arithmetic and elementary algebra appear to be a world apart. Compared to arithmetic, algebraic skill requires another approach to problem solving. Several research projects have reported on these learning difficulties related to algebraic equation solving (Kieran, 1989, 1992; Filloy and Rojano, 1989; Sfard, 1991, 1995; Herscovics and Linchevski, 1994; Linchevski and Herscovics, 1996; Bednarz and Janvier, 1996). Key difficulties mentioned in these reports are constructing equations from word problems, as well as interpreting, rewriting and simplifying algebraic expressions.

In 1995 a project 'Reinvention of algebra' was started at the Freudenthal Institute to investigate which didactical means enable students to make a smooth transition from arithmetic to early algebra. The project resulted in a PhD Thesis (Van Amerom, 2002) that describes two possible ways of approaching the difficulties students have with this transition. The first way, which was strongly influenced by Streefland's interest in students' own productions, is to start from the students' informal strategies and to build more formal methods out of these. How this was researched and which results came out is reported in this paper.



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The second approach, which functioned rather in the background of the project, is to use input from the history of mathematics. This aspect of the project follows recent research on the advantages and possibilities of using and implementing history of mathematics in the classroom (Fauvel and Van Maanen, 2000). Streefland (1996) also emphasized the use of history as a source of inspiration for instructional development. According to him, looking back at the origins makes one better prepared to teach in the future.

ALGEBRA AND ARITHMETIC

Similarities and differences

There is not one generally accepted point of view on what algebra is or how it should be learned, because it has so many applications. But for the sake of practicality, it is useful to distinguish four basic perspectives: (1) algebra as generalized arithmetic, (2) algebra as a problem-solving tool, (3) algebra as the study of relationships, and (4) algebra as the study of structures. Each of these operates in a different medium, where, for example, letters have a specific meaning and role (Usiskin, 1988). In this study we restrict ourselves to linear relationships, formulas and equation solving. The proposed learning activities correspond with the second and third perspectives of algebra as mentioned, and they depend on the dialectic relationship between algebra and arithmetic.

A closer look at the similarities and differences between algebra and arithmetic can help us understand some of the problems that students have with the early learning of algebra. Arithmetic deals with straightforward calculations with known numbers, while algebra requires reasoning about unknown or variable quantities and recognizing the difference between specific and general situations. There are differences regarding the interpretation of letters, symbols, expressions and the concept of equality. For instance, in arithmetic letters are usually abbreviations or units, whereas algebraic letters are stand-ins for variable or unknown numbers. Several researchers (Booth, 1988; Kieran, 1989, 1992; Sfard, 1991) have studied problems related to the recognition of mathematical structures in algebraic expressions. Kieran speaks of two conceptions of mathematical expressions: procedural (concerned to operations on numbers, working towards an outcome) and structural (concerned with operations on mathematical objects). In the 1960s it was already clear that discrepancies between arithmetic and algebra can cause great difficulties in early algebra learning. The difficulty of algebraic language is often underestimated and certainly not self-explanatory. Freudenthal explains that

[i]ts syntax consists of a large number of rules based on principles which, partially, contradict those of everyday language and of the language of arithmetic, and which are even mutually contradictory. (Freudenthal, 1962, p. 35)

He then says:

The most striking divergence of algebra from arithmetic in linguistic habits is a semantical one with far-reaching syntactic implications. In arithmetic 3 + 4 means a problem. It has to be interpreted as a command: add 4 to 3. In algebra 3 + 4 means a number, viz. 7. This is a switch that proves essential as letters occur in the formulae. a + b cannot easily be interpreted as a problem. (ibid.)

The two interpretations (arithmetical and algebraic) of the sum 3 + 4 in the citation above correspond with the terms procedural and structural (or *operational* and *structural*, Sfard, 1991, 1995).

But despite their contrasting natures, algebra and arithmetic also have definite interfaces. For example, algebra relies heavily on arithmetical operations and arithmetical expressions are sometimes treated algebraically. Arithmetical activities like solving addition problems with a missing addend, and the backtracking of operations prepare the students for studying linear relations. Furthermore, the historical development of algebra shows that word problems have always been a part of mathematics that brings together algebraic and arithmetical reasoning.

Cognitive obstacles of learning algebra

An enormous increase in research during the last decade has produced an abundance of new conjectures on the difficult transition from arithmetic to algebra. For instance, with regard to equation solving there is claimed to be a discrepancy known as *cognitive gap* (Herscovics and Linchevski, 1994) or didactic cut (Filloy and Rojano, 1989). These researchers point out a break in the development of operating on the unknown in an equation. Operating on an unknown requires a new notion of equality. In the transfer from a word problem (arithmetic) to an equation (algebraic), the meaning of the equal sign changes from announcing a result to stating equivalence. And when the unknown appears on both sides of the equality sign instead of one side, the equation can no longer be solved arithmetically (by inverting the operations one by one). Kieran (1989) reports that Matz and Davis have done research on students' interpretation of the expression x + 3. Students see this as a process (adding 3) rather than a final result that stands by itself. They have called this difficulty the 'process-product dilemma'. Sfard (1995) compares discontinuities in student conceptions of algebra with the historical development of algebra. She writes that abbreviated (or so-called syncopated) notations in algebra are linked to an operational conception of algebra, whereas symbolic algebra corresponds with a structural conception of algebra.¹ Da Rocha Falcão (1995) suggests that the disruption between arithmetic and algebra is contained in the approach to problem solving. Arithmetical problems can be solved directly, possibly with intermediate answers if necessary. Algebraic problems, on the other hand, need to be translated and written in formal representations first, after which they can be solved. Mason formulates the difference between arithmetic and algebra as follows:

Arithmetic proceeds directly from the known to the unknown using known computations; algebra proceeds indirectly from the unknown, via the known, to equations and inequalities which can then be solved using established techniques. (Mason, 1996, p. 23)

THE LEARNING STRAND

In this study we set out to find out whether a bottom-up-approach (starting from informal methods that students already use) towards algebra can minimize the discrepancy between arithmetic and algebra. In order to do so we designed a learning strand for the primary school grades 5 and 6, and the secondary grade 7. The general topic of the primary school lesson series is recognizing and describing relations between quantities using different representations: tables, sums, rhetoric descriptions and (word) equations. No prior knowledge was required apart from the basic operations and ratio tables. The mathematical content of the student materials includes:

- comparing quantities and equalizing them in different ways;
- discussing conflicts of notations and different meanings of symbols;
- practicing with operations $(+, -, \times, \div)$ and inverting operations;
- problem solving by reasoning with multiple conditions;
- reasoning about expressions (substitution, inversion, and global interpretation).

In the secondary school units these topics were repeated in a compressed way and then extended with solving systems of equations.

In the learning strand the students are challenged to solve pre-algebraic problems many of which were inspired by history. Historically, word problems form an obvious link between arithmetic and algebra. Although algebra has made it much simpler to solve word problems in general, it is remarkable how well specific cases of such mathematical problems were dealt with before the invention of algebra, using arithmetical procedures. Some types of problems are even more easily solved without algebra. The history of algebra has inspired us to design the learning material as follows. Word or story-problems offer ample opportunity for mathematizing activities. Babylonian, Egyptian, Chinese and early Western algebra was primarily concerned with the solving of problems situated in every day life, although these civilizations also showed interest in mathematical riddles and recreational problems. Barter, fair exchange, money, mathematical riddles and recreational puzzles were rich contexts for developing handy solution methods and notation systems and these contexts are also appealing and meaningful for students.

Another possible access is based on notation use, for instance by comparing the historical progress in symbolization and schematization with that of modern students. Rhetorical algebra (written in words) finds itself in between arithmetic and symbolic algebra, so to speak: an algebraic way of thinking about unknowns combined with an arithmetic (procedural) conception of numbers and operations.

The natural preference and aptitude for solving word problems arithmetically forms the basis for the first half of the learning strand, where students' own informal strategies should be adequately included. The transfer to a more formal algebraic approach is instigated by the guided development of algebraic notation, especially the change from rhetorical to syncopated notation, as well as a more algebraic way of thinking. We were interested in determining whether the evolution of intuitive notations used by the learner shows similarities with the historical development of algebraic notation. The barter context in particular appears to be a natural, suitable setting to develop (pre-)algebraic notations and tools such as a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (un)known quantities.

The following Chinese barter problem from the 'Nine Chapters on the Mathematical Art' inspired Streefland (1995, 1996) and the author to use the context of barter as a natural and historically-founded starting point for the teaching of linear equations:

By selling 2 buffaloes and 5 wethers and buying 13 pigs, 1000 qian remains. One can buy 9 wethers by selling 3 buffaloes and 3 pigs. By selling 6 wethers and 8 pigs one can buy 5 buffaloes and is short of 600 qian. How much do a buffalo, a wether and a pig cost?

In modern notation we can write the following system:

 $2b + 5w = 13p + 1000 \quad (1)$ $3b + 3p = 9w \quad (2)$ $6w + 8p + 600 = 5b \quad (3)$ where the unknowns b, w and p stand for the price of a buffalo, a wether and a pig respectively. From the perspective of mathematical phenomenology, Streefland and Van Amerom (1996) posed a number of questions regarding the origin, meaning and purpose of (systems of) equations. The example given above is interesting especially when looking at the second equation, where no number of 'qian' is present. In this 'barter' equation the unknowns b, w and p can also represent the number of animals themselves, instead of their money value. The introduction of an isolated number in the equations (1) and (3) therefore changes not only the medium of the equation (from number of animals to money) but also the meaning of the unknowns (from object-related to quality-of-object-related). Streefland (1995) found in his teaching experiment in which students had to find out how packages can be filled with candies of different prices, that the meaning of literal symbols is an important constituent of the progressive formalization of the pupils. According to Streefland, the students need to be aware of the changes of meaning that letters undergo. In this way, the children's level of mathematical thinking evolves. Streefland mentions the example of asking for the meaning of the equality sign, which indicates 'costs so much', as well as the usual 'is equal to'. These considerations indicate steps in the conceptual development of variables.

We also aimed to investigate how notation and mathematical abstraction are related. The categorization rhetorical – syncopated – symbolic is the result of the current conception of how algebra developed, and for this reason it is often mistaken for a gradation of mathematical abstraction (Radford, 1997). When the development of algebra is seen from a sociocultural perspective instead, syncopated algebra was not an intermediate stage of maturation but it was merely a technical matter. As Radford explains, the limitations of writing and lack of book printing quite naturally led to abbreviations and contractions of words. Students may reveal similar needs for efficiency when they develop their own notations (from contextbound notation to an independent, general mathematical language), but this need not coincide with the conceptual development of letter use.

CLASSROOM EXAMPLES

The field testing of the units of the learning strand was done in two rounds. In the first round the materials were tried out in two primary classes each consisting of 30 grade 5–6 students (10–12 year olds). In the second round the revised materials were proposed to three grade 6 primary classes and two grade 7 secondary classes (12–13 year olds). During the trial data were collected in several ways: observing lessons, gathering students notes, ad-

ministrating post-tests, and having questionnaires completed by students and teachers.

In this paper we will focus on the issues of symbolizing and reasoning by taking one case from the first round situated in grade 5–6 and another case from the second round in grade 7.

Case 1: Symbolizing and the meaning of letters

Evoking shortened notations forms one of the spear points in the learning strand. The introduction of shortened notations in grade 5–6 is done in the context of a game of cards. In this activity the students work with cards displaying the scores of each round. There are four types of representations: a description of what happens in each round, the scores of all the players in points, a description of a relation between the scores (for example, 'Petra has 5 points more than Jacqueline') and a word formula expressing a certain relation ('points Petra = points Jacqueline + 5'). The children are asked to find the right cards for every round, and then design their own cards for two more rounds of playing cards. Sometimes there is more than one card to describe the scores, when the scores are related in two different ways (for example, 'twice as much' and '5 more'), in which case a switch of perspective is needed. The following classroom vignette shows how a pupil carried out the task of shortening notations too rigorously (Van Amerom, 2002, p. 132).

Classroom vignette 1

Robert:	Points Petra plus Anton is Jacqueline.
Teacher:	Something is not right there Well, maybe the calculation is.
Tim:	Points Petra plus points Anton.
Teacher:	Indeed! You have shortened too much. It is all about the points of these people. You can't just add people to the points that they get, that's impossible!
(Laughing in the	class.)
Teacher:	It is actually about the number of points: the number of points that Petra has plus the number of points that Anton has is the number to points that Jacqueline has.
Renske:	That's what I had!
Teacher:	Yes, we talked about that for a little bit yesterday. It is not wrong what you say, Robert, but it is not clear when it is too short. It is important that you say it clearly.

In this way the pupils and the teacher ascertained together that when you use abbreviations, it must be clear to everyone what you mean. At the same time one can wonder if it is necessary to be so precise at this stage, because the pupils realize exactly what the letters are about. Therefore a compromise should be found between precise and unambiguous notations on one hand, and intuitive (probably inconsistent) productions of children on the other.

Sometimes the teacher let a good opportunity pass by, as shown in classroom vignette 2. In one of the lessons children suggested what could be the meaning of the expression ' $pA = 3 \times pJ$ '. Our decision to use this kind of symbolism is based on other pupils' free productions in a preliminary try-out. The letter combination maintains the link with the context: the letter *p* stands for 'number of points belonging to' and the capital letter stands for the person in question, in this case Annelies and Jeroen. In the expression, such a letter combination behaves like a variable for which numbers can be substituted. When the score of one of the players is given, the expression becomes an equation that can be solved. The teacher asked the children for an example that will illustrate that the relation between the variables *pA* and *pJ* is '3 times as much'.

Classroom vignette 2

If we think of points, what would be possible? You have to write it down in a handy way, just like in <i>Pocket Money</i> , which numbers are possible.
3 and 9.
Who has 3 and who has 9?
Annelies has 9.
How would you write it down? Why don't you show us on the blackboard.
A - 9 p j - 3 p
I would write an equal sign, not a line.

The teacher missed a good opportunity to discuss three samples of inconsistent symbolizing: Yvette's choice to write a capital letter A and then a small letter j, her use of the letter p as a unit even though it is already part of the variable, and Sanne's suggestion at the end. Apparently it was not a problem to the children that letters mean different things at the same time. As long as the pupils and the teacher are all conscious of this fact, the development and refinement of notations is a natural process. On the other hand, it is not our intention to cause unnecessary confusion regarding the meaning of letters. It was decided that if children have a natural tendency to use the letter p as a unit, it should not be included in the expressions and formulas.

The lesson materials were adjusted and tested again in 1999. The dual character of the learning strand – to develop reasoning and symbolizing

1 X ROOS 11 15 0 85 10xR + 5xF = 15,75 5xR+10xF=14,25 er Komen Krozen bij en er goan 5 floxen af dat verschil is 1,50. er komen i roos bij en er goal i floxaf. dat verschil is 0,30 eeste 15 Roten: 15425 17,25:15 :11,15×10: 4,25:5:0,85.

Figure 1. Symbolic notations and informal reasoning.

skills in the study of relations – was maintained but placed in a more problem-oriented setting and with a more explicit historical component at primary and secondary level.

Case 2: Symbolizing versus reasoning

Another important issue to be studied was the students' reasoning in connection with their symbolizing. In the following we will discuss two examples of student work from the second round of the classroom experiment to demonstrate that (pre-)algebraic symbolizing tends to be more difficult for students than reasoning.

Encouraged by the ideas and results of the classroom experiment on candy by Streefland (1995), a grade 7 unit on equation solving was designed based on mathematizing of fancy fair attractions into equations. One of the tasks in the written test was:

Sacha wants to make two bouquets using roses and phloxes. The florist replies: "Uhm ... 10 roses and 5 phloxes for 15,75, and 5 roses and 10 phloxes for 14,25; that will be 30 guilders altogether please." What is the price of one rose? And one phlox? Show your calculations.

One of the outcomes of the experiment is that algebraic equation solving need not necessarily develop synchronously with algebraic symbolization.



Figure 2. Informal notations and algebraic reasoning.

For instance, we have observed student work where a correct symbolic system of equations was followed by an incorrect or lower order strategy, or where the student proceeded with the solution process rhetorically. The student whose work is shown in Figure 1 (Van Amerom, 2002, p. 230) mathematizes the problem by constructing a system of equations, and then applies an informal, pre-algebraic exchange strategy which is developed in the unit. Below the equations she writes: "We get 5 roses more and 5 phloxes less, the difference is 1.50. We get 1 rose more and 1 phlox less, the difference is 0.30." The calculations show that she continues the pattern to get 15 roses for the price of 17.25 guilders, and then she determines the price of 1 rose and 1 phlox. The level of symbolizing may appear to be high at first due to the presence of symbolic equations, but the student does not operate on the equations. The equations may have helped her structure the problem but they are not a part of the solution process. And even though the unknown numbers of flowers are an integral part of her reasoning, the letters representing them are not needed in the calculations. There is a parallel here with the historical development of symbolizing the solution. In the rhetorical and syncopated stages of algebra the unknown was mentioned only at the start and at the end of the problem; the calculations were done using only the coefficients.

Alternatively the solution in Figure 2 (Van Amerom, 2002, p. 227) illustrates how the level of reasoning can be higher than the level of symbolizing. This student solves the system of equations

 $2 \times h + 2 \times k = 66$ $3 \times h + 4 \times k = 114$ by doubling the first equation and then subtracting the second from it. First he deals with the right hand sides of the equations $(66 \times 2 \text{ and } 132 - 114)$. In between the two horizontal lines we observe how he multiplies the terms with the unknowns. Then he writes: "But the task says 3h so 18 is 1 h." Finally he substitutes the value 18 to solve for k. A remarkable contrast presents itself. This student successfully applies a formal algebraic strategy of eliminating one unknown by operating on the equations, while his symbolizing is still at a very informal level. Again the unknown is only partially included in the solution process; it appears only where necessary. In other words, both examples of equation solving illustrate that competence of reasoning and symbolizing are separate issues.

EPILOGUE

Difficulties in the learning of algebra can be partially ascribed to ontological differences between arithmetic and algebra. Our study uses informal, pre-algebraic methods of reasoning and symbolizing as a way to facilitate the transition from an arithmetical to an algebraic mode of problem solving. We have shown some examples in which informal notations deviate from conventional algebra syntax, such as the inconsequent use of letters and the pseudo-absence of the unknown in solving systems of equations. These side effects bring new considerations for teaching: how can we bridge the gap between students' intuitive, meaningful notations and the more formal level of conventional symbolism? The observation that symbolizing and reasoning competencies are not necessarily developed at the same pace – neither in ancient nor in modern times – also has pedagogical implications. It appears that equation solving does not depend on a structural perception of equations, nor does it rely on correct manipulations of the equation.

In retrospect we can say that knowledge of the historical development of algebra has led to a sharper analysis of student work and the discovery of certain parallels between contemporary and historical methods of symbolizing. Streefland's notice to look back at the origins in order to anticipate has turned out to be a valuable piece of his legacy.

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NOTE

1. According to Struik (1990) this classification of algebra problems into rhetorical, syncopated and symbolic algebra appeared firstly in Nesselmann, *Die Algebra der Griechen*, Berlijn 1842.

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Freudenthal Institute, Utrecht University, Aïdadreef 12, 3561 GE Utrecht, The Netherlands Telephone +31 (0)302635555, Fax +31 (0)302660430, E-mail: b.vanamerom@fi.uu.nl